Les Houches Summer School, Session CVII--
Current Trends in Atomic Physics

ÉCOLE DE PHYSIQUE des HOUCHES

UNIVERSITÉ Grenoble Alpes

Quantum Control, Measurement and Tomography

Lecture II: Quantum Measurement

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University of New Mexico

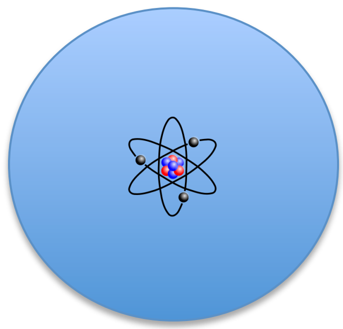
http://www.cquic.org/

Syllabus

- Lecture I: Quantum Control
  - Foundations of Quantum Control Theory
  - Atomic Spins as a Quantum Control Testbed
- Lecture II: Quantum Measurement
  - Foundation of Quantum Measurement Theory
  - Continuous measurement and quantum trajectories
- Lecture III: QND Measurement – Spin Squeezing
  - Measuring Spins via Polarization Spectroscopy
  - Quantum control for enhanced spin squeezing
- Lecture IV: Quantum Tomography
  - Foundation of Quantum Tomography
  - Quantum Tomography via Continuous Measurement

Quantum Measurement Theory

Textbook Quantum Measurement



Quantum system

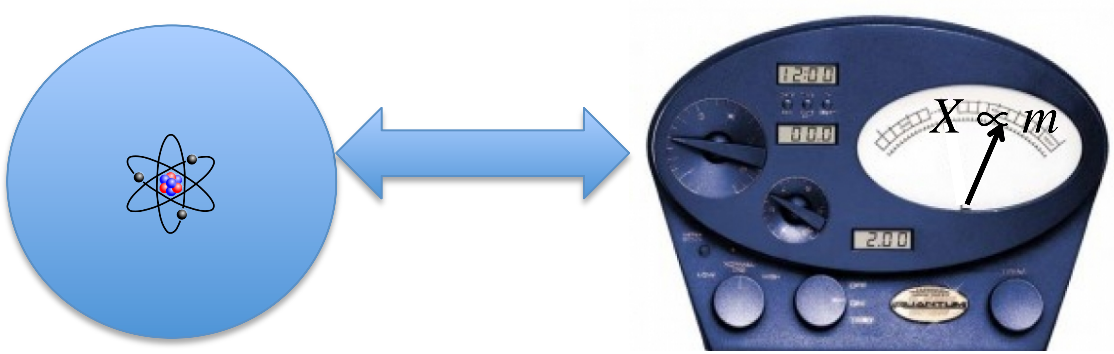
$$|\psi\rangle_S = \sum_m c_m |m\rangle_S$$

"Classical Meter"

Meter measures observable

$$\hat{O} = \sum_m m |m\rangle\langle m|$$

Textbook Quantum Measurement



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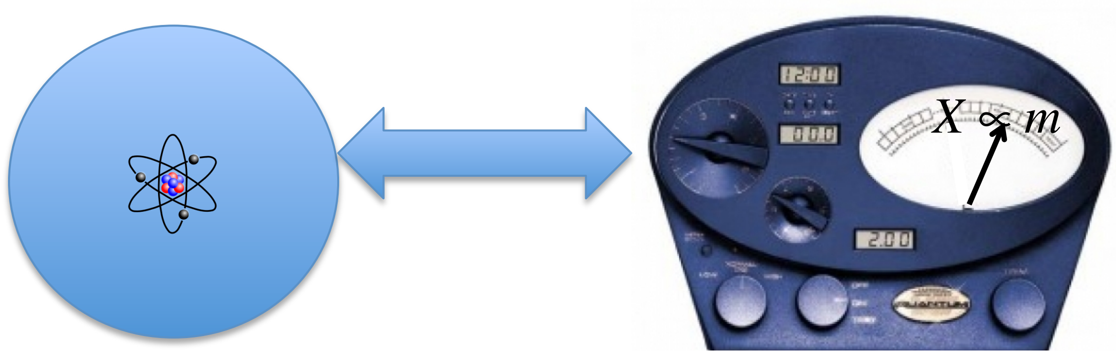


"Classical Meter"

Quantum system

Meter measures observable

$$\hat{O} = \sum_m m |m\rangle\langle m|$$

Post-Measurement $$|\psi\rangle_S \Rightarrow |m\rangle\langle m||\psi\rangle_S = c_m |m\rangle_S$$

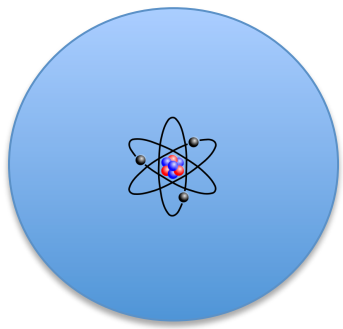
Renormalize $$|\psi\rangle_S = |m\rangle_S$$

"Collapse of the wave function"

Probability of finding m: $$P_m = |\langle m|\psi\rangle|^2 = |c_m|^2$$

Born Rule

Von Neumann Theory of Measurement



“Quantum degrees of freedom"

Quantum system

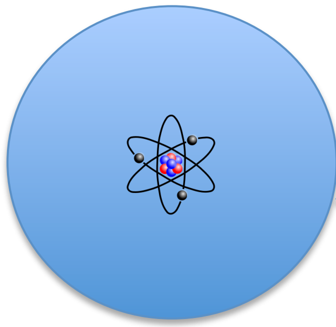
$$|\psi\rangle_S = \sum_m c_m |m\rangle_S$$

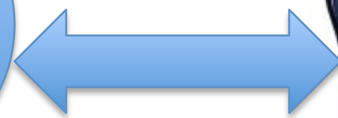
Quantum Meter (ancilla)

$$|\Phi_0\rangle_A$$

$$|\Psi\rangle_{SA}^{in} = |\psi\rangle_S \otimes |\Phi_0\rangle_A$$

Von Neumann Theory of Measurement





$$\hat{U}_{SA} = e^{-i\chi \hat{O}_s \otimes \hat{P}_A}$$

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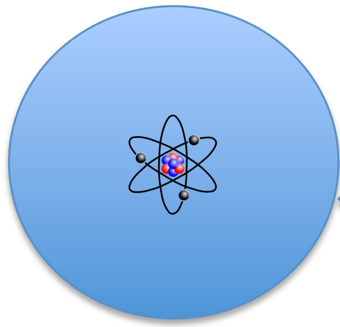

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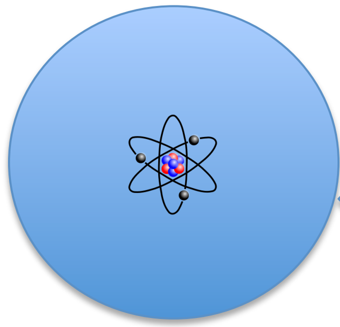

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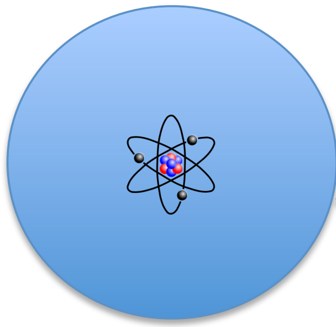
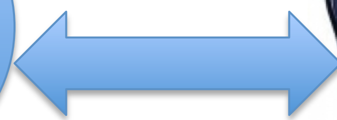
Von Neumann Theory of Measurement





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Quantum system

$$|\psi\rangle_S = \sum_m c_m |m\rangle_S$$

Quantum Meter (ancilla)

$$|\Phi_0\rangle_A$$

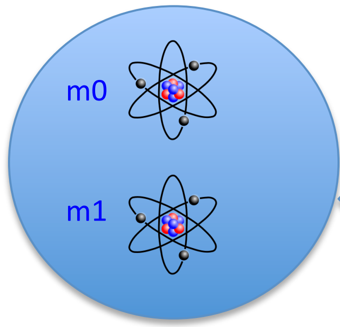
$$|\Psi\rangle_{SA}^{out} = \hat{U}_{SA}|\Psi\rangle_{SA}^{in} = \hat{U}_{SA}|\psi\rangle_S \otimes |\Phi_0\rangle_A = e^{-i\chi\hat{O}_s\otimes\hat{P}_A}|\psi\rangle_S \otimes |\Phi_0\rangle_A =$$

$$= \sum_m c_m |m\rangle_S \otimes e^{-i\chi m\hat{P}_A}|\Phi_0\rangle_A = \sum_m c_m |m\rangle_S \otimes |\Phi_{\chi m}\rangle_A$$

Meter is displaced by an amount proportional to the eigenvalue to be measured.

Von Neumann Theory of Measurement

Example:
Atom is in superposition of two eigenstates



$$\hat{U}_{SA} = e^{-i\chi \hat{O}_s \otimes \hat{P}_A}$$

$$\left|\Psi\right\rangle_{SA}^{out} = \hat{U}_{SA}\left|\Psi\right\rangle_{SA}^{in} = \hat{U}_{SA}\left|\Psi\right\rangle_{S} \otimes \left|\Phi_0\right\rangle_A = \sum_m c_m e^{-i\chi \hat{O}_s \otimes \hat{P}_A}\left(\left|m\right\rangle_s \otimes \left|\Phi_0\right\rangle_A\right)$$

$$= \sum_m c_m \left|m\right\rangle_s \otimes e^{-i\chi m \hat{P}_A}\left|\Phi_0\right\rangle_A = \sum_m c_m \left|m\right\rangle_s \otimes \left|\Phi_{\chi m}\right\rangle_A$$

$$= c_0\left|m_0\right\rangle_s \otimes \left|\Phi_{\chi m_0}\right\rangle_A + c_1\left|m_1\right\rangle_s \otimes \left|\Phi_{\chi m_1}\right\rangle_A$$

If $\left\langle \Phi_{\chi m'} \middle| \Phi_{\chi m}\right\rangle = \delta_{mm'}$ then meter states are perfectly distinguishable → **Projection**

"Find" meter in state $\left|\Phi_{\chi m}\right\rangle_A$ → Collapse system state as well

Post-Measurement (Unnormalized)

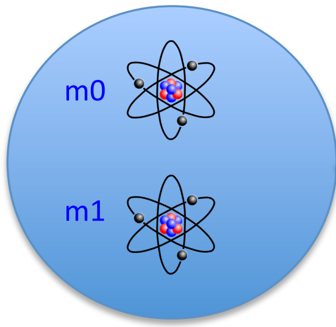
$$\left|\tilde{\psi}\right\rangle_S^{out} = \left\langle \Phi_{\chi m}\right| U_{SA}\left|\Psi\right\rangle_{SA}^{in} = c_m\left|m\right\rangle_S \; = \left(\left|m\right\rangle\left\langle m\right|\right)\left|\Psi\right\rangle_S^{in}$$

Probability

$$P_{\chi m} = \left\| \left|\tilde{\psi}_s^{out}\right\rangle \right\|^2 = \left|c_m\right|^2$$

General Theory of Measurement

Example:
Atom is in
superposition of
two eigenstates



Weaker coupling

$$\hat{U}_{SA} = e^{-i\chi \hat{O}_s \otimes \hat{P}_A}$$

$$\left|\Psi\right\rangle_{SA}^{out} = \hat{U}_{SA}\left|\Psi\right\rangle_{SA}^{in} = \hat{U}_{SA}\left|\Psi\right\rangle_S \otimes \left|\Phi_0\right\rangle_A = \sum_m c_m e^{-i\chi \hat{O}_s \otimes \hat{P}_A}\left(\left|m\right\rangle_s \otimes \left|\Phi_0\right\rangle_A\right)$$

$$= \sum_m c_m \left|m\right\rangle_s \otimes e^{-i\chi m \hat{P}_A}\left|\Phi_0\right\rangle_A = \sum_m c_m \left|m\right\rangle_s \otimes \left|\Phi_{\chi m}\right\rangle_A$$

$$= c_0\left|m_0\right\rangle_s \otimes \left|\Phi_{\chi m_0}\right\rangle_A + c_1\left|m_1\right\rangle_s \otimes \left|\Phi_{\chi m_1}\right\rangle_A$$

If $\left\langle \Phi_{\chi m'} \middle| \Phi_{\chi m} \right\rangle \neq \delta_{mm'}$ then meter states are not perfectly distinguishable → **POVM**

Example: Measure meter observable X→ (weak) backaction on system state

Unnormalized

$$\left|\tilde{\psi}\right\rangle_S^{in}\Big|_X = \left\langle X_A \middle| \hat{U}_{SA} \middle| \Psi \right\rangle_{SA}^{in} = \hat{K}_X \left|\psi\right\rangle_S^{in}$$

Krause operator

$$\hat{K}_X = \left\langle X_A \middle| \hat{U}_{SA} \middle| \Phi_{0,A} \right\rangle$$

General Theory of Measurement

Post-measurement state

$$\left|\tilde{\psi}\right\rangle_S^{out}\Big|_X = \left\langle X_A \middle| \hat{U}_{SA} \middle| \Psi \right\rangle_{SA}^{in} = \hat{K}_X \left|\psi\right\rangle_S^{in}$$

Krause operator

$$\hat{K}_X = \left\langle X_A \middle| \hat{U}_{SA} \middle| \Phi_{0,A} \right\rangle$$

Probability of finding X on the meter

$$P_X = \left\| \left|\tilde{\psi}_s^{out}\right|_X \right\|^2 = \left\langle \psi_s^{in} \middle| \hat{K}_X^\dagger \hat{K}_X \middle| \psi_s^{in} \right\rangle = \left\langle \psi_s^{in} \middle| \hat{E}_X \middle| \psi_s^{in} \right\rangle$$

$$\left\{ \hat{E}_X = \hat{K}_X^\dagger \hat{K}_X \right\}$$

POVM = Positive Operator Valued Measure

- Positive operators: $P_{X|\psi} = \left\langle \psi \middle| \hat{E}_X \middle| \psi \right\rangle \geq 0, \qquad P_{X|\rho} = Tr\left(\hat{\rho}\hat{E}_X\right) \geq 0$
- Completeness: $\int dX\, \hat{E}_X = \hat{I} \Rightarrow \int dX\, P_{X|\rho} = 1$

$$\int dX\, \hat{E}_X = \int dX\, \hat{K}_X^\dagger \hat{K}_X = \int dX\, \left\langle \Phi_0 \middle| \hat{U}_{SA}^\dagger \middle| X \right\rangle \left\langle X \middle| \hat{U}_{SA} \middle| \Phi_0 \right\rangle = \left\langle \Phi_0 \middle| \hat{U}_{SA}^\dagger \hat{U}_{SA} \middle| \Phi_0 \right\rangle = 1$$

General Theory of Measurement

Most general measurement in quantum mechanics

$\left\{\hat{E}_\mu\right\}$ POVM = Positive Operator Valued Measure

- Positive operators: $\hat{E}_\mu \geq 0,$
- Completeness: $\sum_\mu \hat{E}_\mu = \hat{I}$
- Born rule: $P_\mu = Tr\left(\hat{E}_\mu \hat{\rho}\right)$

- Beyond projective measurements onto eigenstates of observables
- Number of measurement outcomes can be arbitrary
- Post-measurement state depends on measurement model

Example: Measurement with Gaussian Noise

Ancilla State (initial state of the meter):

$$\langle X | \Phi \rangle_A = \pi^{-1/4} e^{-\frac{1}{2}X^2}$$

Gaussian centered on X=0
Unit variance in $\left|\langle X | \Phi \rangle_A\right|^2$
Unnormalized

Kraus Operator

$$\hat{K}_X = \langle X_A | \hat{U}_{SA} | \Phi_{0,A} \rangle = \langle X_A | e^{-i\chi \hat{O}_S \otimes \hat{P}_A} | \Phi_{0,A} \rangle$$

$$= \langle X_A - \chi \hat{O}_s | \Phi_{0,A} \rangle = \pi^{-1/4} e^{-\frac{1}{2}\left(X - \chi \hat{O}_s\right)^2} = \pi^{-1/4} e^{-\frac{\chi^2}{2}(\hat{O}_s - \underbrace{X/\chi}_{\mathcal{M}})^2}$$

$$\hat{K}_{\mathcal{M}} \equiv (\chi^2 \pi)^{-1/4} e^{-\frac{\chi^2}{2}\left(\hat{O}_s - \mathcal{M}\right)^2}$$

Recall $\hat{O}_s = \sum_m m |m\rangle\langle m|$

POVM Elements

$$\hat{E}_{\mathcal{M}} = \hat{K}^\dagger_{\mathcal{M}} \hat{K}_{\mathcal{M}} = \frac{e^{-\chi^2\left(\hat{O}_s - \mathcal{M}\right)^2}}{\sqrt{\pi\chi^2}} = \sum_m \frac{e^{-\chi^2(m - \mathcal{M})^2}}{\sqrt{\pi\chi^2}} |m\rangle\langle m|$$

Projector on *m*, convolved with a noisy Gaussian

Example: Measurement with Gaussian Noise

Probability of finding X on the meter

$$P_{\mathcal{M}} = \langle \psi_s^{in} | \hat{E}_{\mathcal{M}} | \psi_s^{in} \rangle$$

$$\hat{E}_{\mathcal{M}} = (\chi^2 \pi)^{-1/2} \sum_m e^{-\chi^2 (m-\mathcal{M})^2} |m\rangle\langle m|$$

Note:

$$\lim_{\chi\to\infty} \hat{E}_X = \sum_m \delta(m-\mathcal{M}) |m\rangle\langle m| = |m=\mathcal{M}\rangle\langle m=\mathcal{M}|$$

Projector!

If system is in eigenstate $|m\rangle$ $P_{\mathcal{M}|m} = \langle m | \hat{E}_{\mathcal{M}} | m\rangle = (\chi^2\pi)^{-1/2} e^{-\chi^2(m-\mathcal{M})^2}$

Measurement is uncertain since meter is quantum object with fluctuations

Uncertainty in deduced m-value due to quantum meter: $\Delta m^2\big|_{meter} = \chi^{-2}$ (meter noise)

If system not in eigenstate $|\psi\rangle_S = \sum_m c_m |m\rangle_S$ $\Delta m^2\big|_{state} = \sum_m (m - \langle m\rangle)^2 |c_m|^2$

(Projection noise)

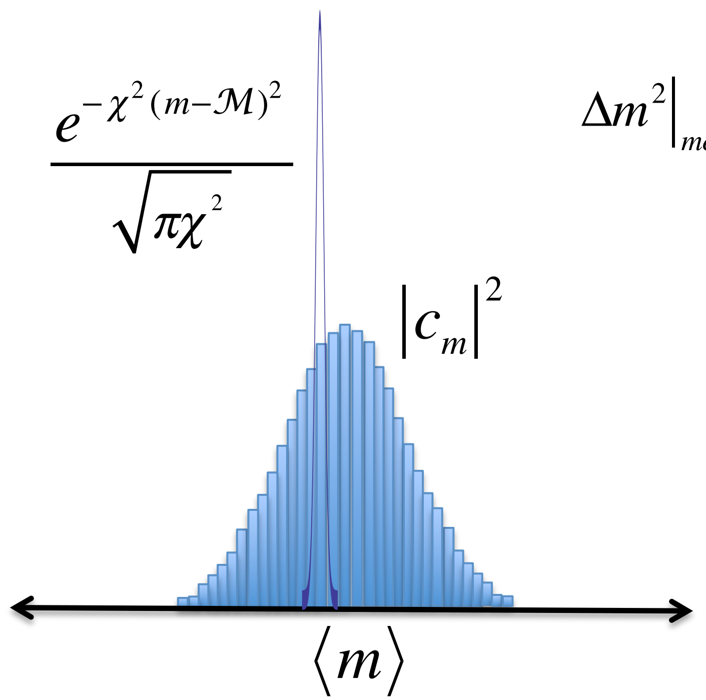
$$P_{\mathcal{M}|m} = \langle m | \hat{E}_{\mathcal{M}} | m\rangle = (\chi^2\pi)^{-1/2} \sum_m e^{-\chi^2(m-\mathcal{M})^2} |c_m|^2$$

Measurement Backaction

Post-Measurement State (Unnormalized)

$$\left|\tilde{\psi}\right\rangle_S^{out}\Big|_{\mathcal{M}} = \hat{K}_{\mathcal{M}}\left|\psi\right\rangle_S^{in} = (\chi^2\pi)^{-1/4}\sum_m e^{-\frac{\chi^2}{2}(m-\mathcal{M})^2} c_m \left|m\right\rangle$$

$$\Delta m^2\Big|_{meter} << \Delta m^2\Big|_{state}$$



Measurement Backaction

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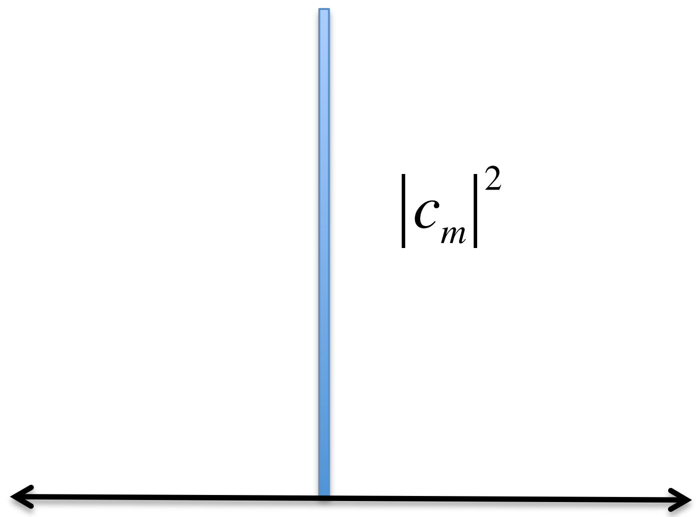

Projective Measurement

Measurement Backaction

Post-Measurement State (Unnormalized)

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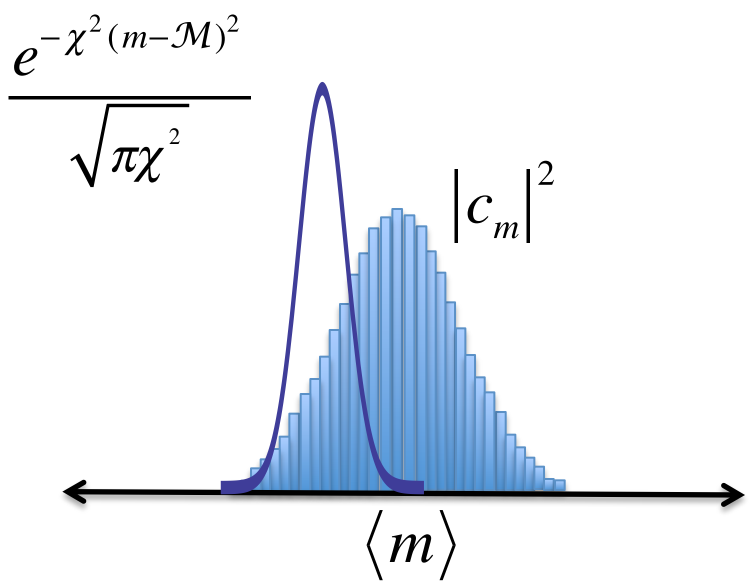

Measurement Backaction

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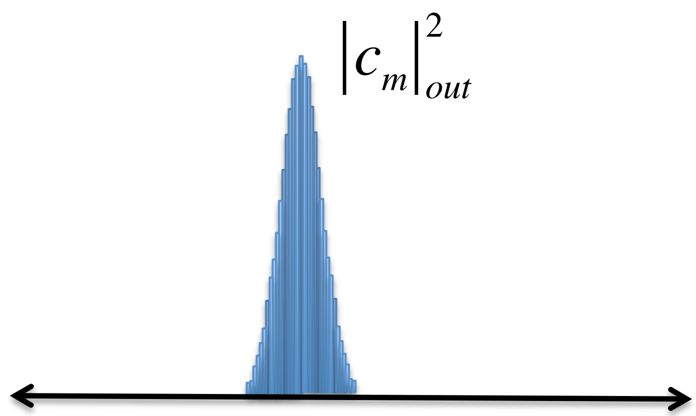


Weak measurement

(Do not confuse with "weak value" a la Aharonov)

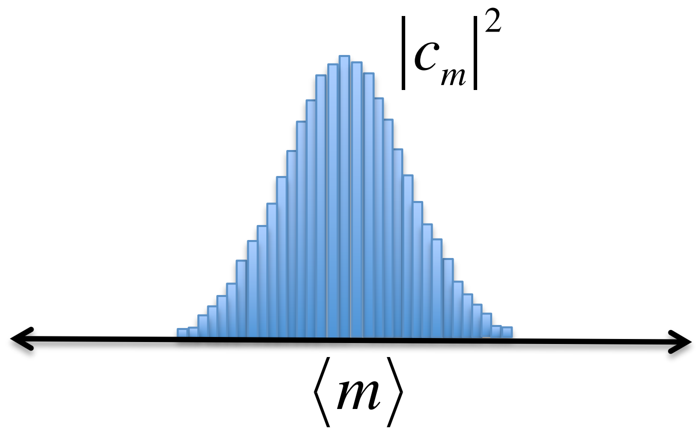
Measurement Backaction

Post-Measurement State (Unnormalized)

$$\left|\tilde{\psi}\right\rangle_S^{out}\Big|_{\mathcal{M}} = \hat{K}_{\mathcal{M}}\left|\psi\right\rangle_S^{in} = \pi^{1/4}\sum_m e^{-\frac{\chi^2}{2}(m-\mathcal{M})^2} c_m \left|m\right\rangle$$

$$\Delta m^2\Big|_{meter} >> \Delta m^2\Big|_{state}$$

$$\frac{e^{-\chi^2(m-\mathcal{M})^2}}{\sqrt{\pi\chi^2}}$$



Backaction negligible

Measurement Backaction

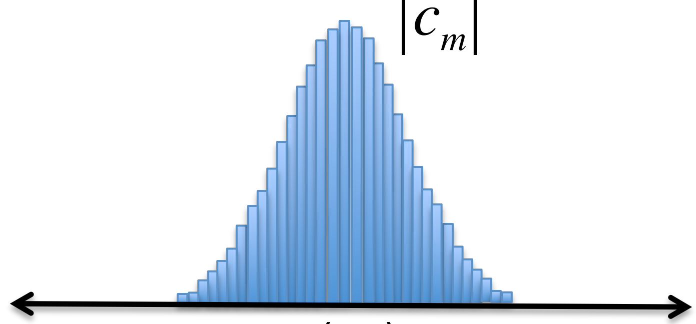
Post-Measurement State (Unnormalized)

$$\left|\tilde{\psi}\right\rangle_S^{out}\Big|_{\mathcal{M}} = \hat{K}_{\mathcal{M}}\left|\psi\right\rangle_S^{in} \approx \pi^{1/4} e^{-\frac{\chi^2}{2}(\langle m\rangle-\mathcal{M})^2}\sum_m c_m|m\rangle$$

$$\Delta m^2\big|_{meter} >> \Delta m^2\big|_{state}$$

$$\frac{e^{-\chi^2(m-\mathcal{M})^2}}{\sqrt{\pi\chi^2}}$$

$$\left|c_m\right|^2$$



$$\langle m\rangle$$

Backaction negligible

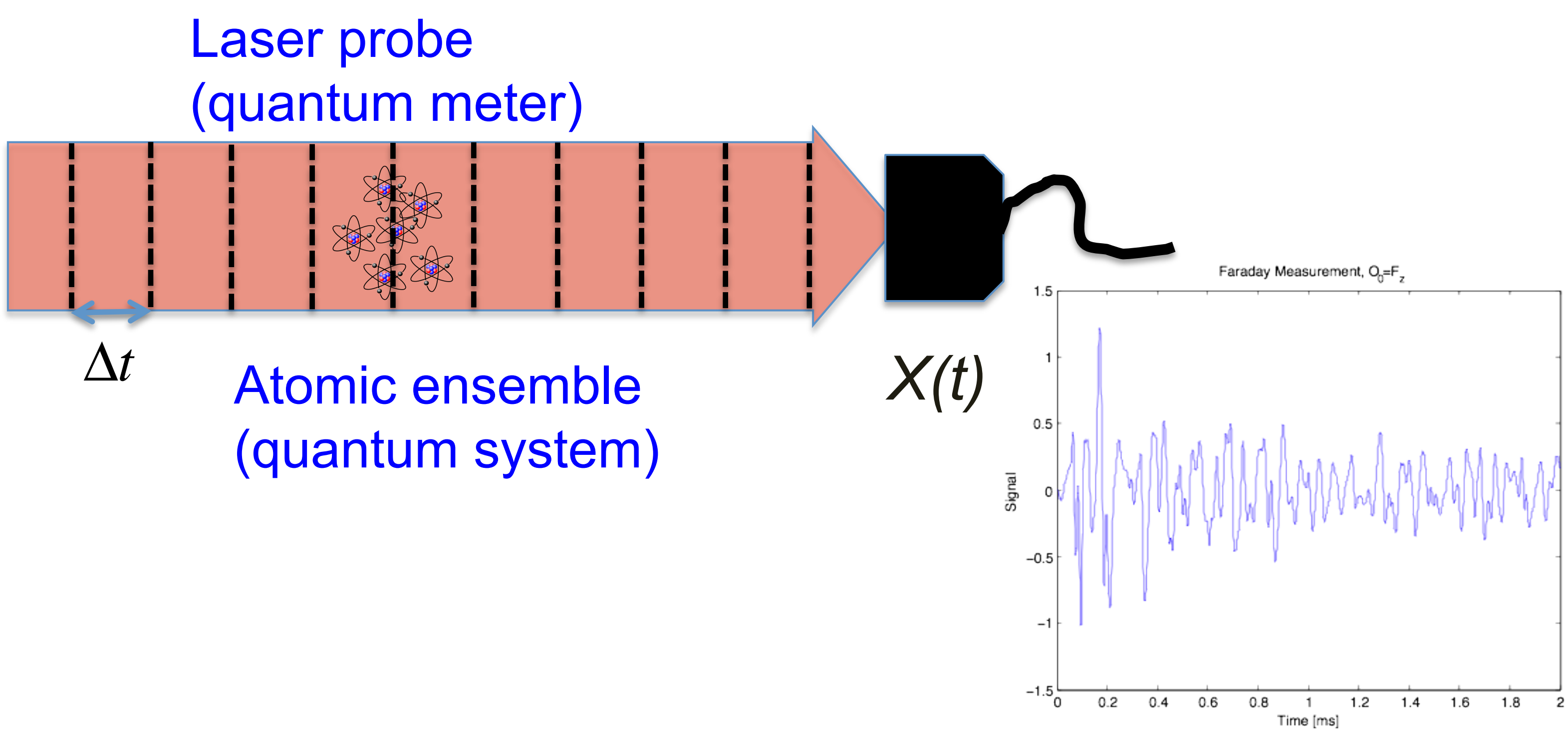
Normal random
0=mean, unit variance

$$P_{\mathcal{M}} = \left\langle\psi_s^{in}\right|\hat{E}_{\mathcal{M}}\left|\psi_s^{in}\right\rangle \approx \left(\pi\chi^2\right)^{-1/2} e^{-\chi^2\left(\langle\hat{O}_s\rangle-\mathcal{M}\right)^2}$$

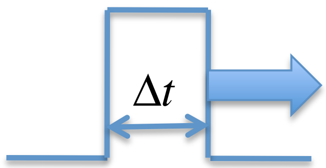
$$\mathcal{M} = \left\langle\psi_s\right|\hat{O}_s\left|\psi_s\right\rangle + \underbrace{\frac{1}{\sqrt{2}\chi}\mathcal{N}}_{Shot\ Noise}$$

Mean field approximation.

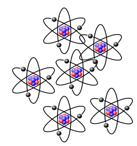
Continuous Measurement



Entangling Interaction

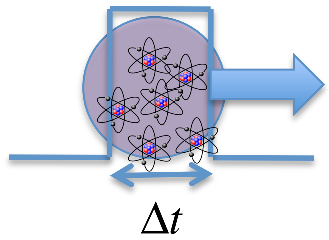


Light packet
(quantum ancilla)

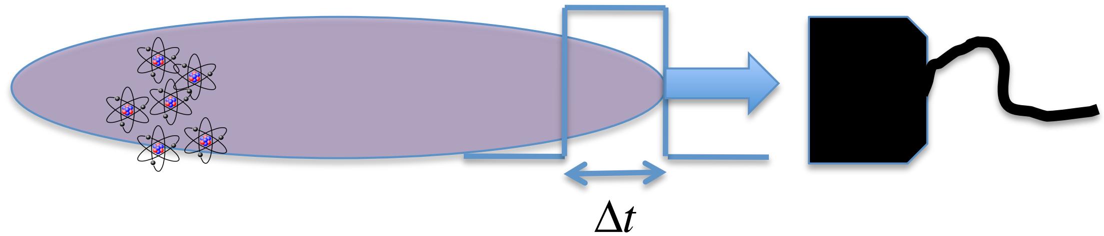


Atomic ensemble
(quantum system)

Entangling Interaction



Entangling Interaction



Signal $\propto \Delta t$ Shot noise $\propto \sqrt{\Delta t}$

SNR $\propto \dfrac{1}{\sqrt{\Delta t}}$ SNR variance $= \dfrac{1}{\kappa \Delta t} = \dfrac{1}{2\chi^2}$

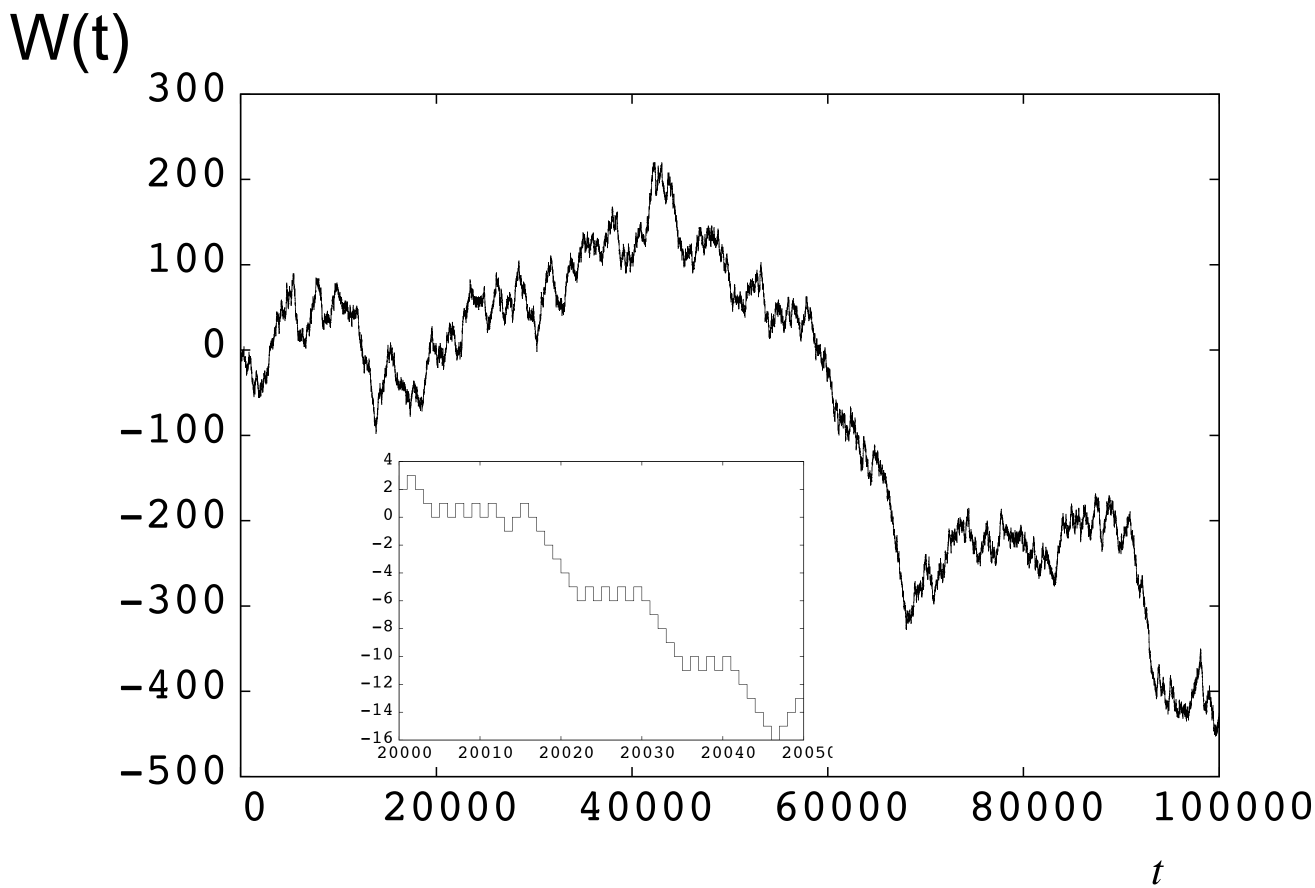
κ = measurement rate

As $\Delta t \to 0$, weak measurement in any time-slice

$$\mathcal{M}(t) = \langle \psi(t) | \hat{O} | \psi(t) \rangle + \frac{1}{\sqrt{\kappa \Delta t}} \mathcal{N}(t)$$

White Noise: Wiener Stochastic Process

Brownian motion – random walk in 1D



$$P(W(t)) = \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{W^2}{2t} \right\}$$

Gaussian:

$$\overline{W(t)} = 0 \qquad \overline{W^2(t)} = t$$

White Noise: Wiener Stochastic Process

Wiener interval $$\Delta W(t) = W(t+\Delta t) - W(t)$$

$$\langle \Delta W(t_1)\Delta W(t_2)\rangle = \begin{cases} 0; & |t_1 - t_2| \geq \Delta t \\ \Delta t - |t_1 - t_2|; & |t_1 - t_2| \leq \Delta t \end{cases}$$

Ito stochastic differential $$dW(t) = \lim_{\Delta t\to 0} \Delta W(t) \sim \sqrt{dt}\,\mathcal{N}(t)$$

$$\overline{dW(t)} = 0 \qquad \left(dW(t)\right)^2 = dt \qquad dW(t)dt = 0$$

White Noise $$\xi(t) = \lim_{\Delta t\to 0} \frac{\Delta W(t)}{\Delta t} = \frac{dW(t)}{dt} \qquad \overline{\xi(t)\xi(t')} = \delta(t-t')$$

Continuous Measurement $$\mathcal{M}(t) = \langle \hat{O} \rangle + \lim_{\Delta t\to 0} \frac{1}{\sqrt{\kappa \Delta t}}\mathcal{N}(t) = \langle \hat{O} \rangle + \frac{1}{\sqrt{\kappa}}\frac{dW(t)}{dt}$$

Stochastic Schrödinger Equation

Evolution of the state conditioned on the measurement record

$$\left|\psi_c(t+dt)\right\rangle = \frac{\hat{K}_{\mathcal{M}}(t,dt)\left|\psi_c(t)\right\rangle}{\left\|\hat{K}_{\mathcal{M}}(t,dt)\left|\psi_c(t)\right\rangle\right\|}$$

Kraus operator for continuous measurement

$$\hat{K}_{\mathcal{M}}(t,dt)\left|\psi_c(t)\right\rangle = e^{-\frac{\kappa dt}{4}\left(\hat{O}-\mathcal{M}(t)\right)^2}\left|\psi_c(t)\right\rangle$$

$$\mathcal{M}(t) = \left\langle\psi(t)\right|\hat{O}\left|\psi(t)\right\rangle + \frac{1}{\sqrt{\kappa}}\frac{dW(t)}{dt}$$

$$\hat{K}_{\mathcal{M}}(t,dt) = e^{-\frac{\kappa dt}{4}\left(\Delta\hat{O}-dW/\sqrt{\kappa}dt\right)^2} \qquad \Delta\hat{O} \equiv \hat{O} - \left\langle\psi(t)\right|\hat{O}\left|\psi(t)\right\rangle$$

Stochastic Schrödinger Equation

$$\hat{K}_{\mathcal{M}}(t,dt) = \exp\left\{-\frac{\kappa dt}{4}\left(\Delta\hat{O}(t) - \frac{1}{\sqrt{\kappa}}\frac{dW}{dt}\right)^2\right\}$$

$$= \exp\left\{-\frac{\kappa dt}{4}\Delta\hat{O}^2(t) + \frac{\sqrt{\kappa}dW}{2}\Delta\hat{O}(t)\right\}$$

$$= 1 - \frac{\kappa dt}{4}\Delta\hat{O}^2(t) + \frac{\sqrt{\kappa}dW}{2}\Delta\hat{O}(t) + \frac{1}{2}\frac{\kappa dW^2}{4}\Delta\hat{O}^2(t)$$

$$\hat{K}_{\mathcal{M}}(t,dt) = 1 - \frac{\kappa}{8}\Delta\hat{O}^2(t)dt + \frac{\sqrt{\kappa}}{2}\Delta\hat{O}(t)dW$$

$$\langle\psi|\hat{K}^\dagger\hat{K}|\psi\rangle = \langle\psi|1 - \frac{\kappa}{4}\Delta\hat{O}^2(t)dt + \left(\frac{\sqrt{\kappa}}{2}\Delta\hat{O}(t)dW\right)^2 + \sqrt{\kappa}\Delta\hat{O}(t)dW|\psi\rangle = 1$$

Stochastic Schrödinger Equation

$$\left|\psi_c(t+dt)\right\rangle = \hat{K}(t,dt)\left|\psi_c(t)\right\rangle$$

$$= \left(1 - \frac{\kappa}{8}\Delta\hat{O}^2(t)dt\right)\left|\psi_c(t)\right\rangle + \frac{\sqrt{\kappa}}{2}\Delta\hat{O}(t)dW\left|\psi_c(t)\right\rangle$$

$$\left|d\psi_c(t)\right\rangle = \left|\psi_c(t+dt)\right\rangle - \left|\psi_c(t)\right\rangle$$

$$\left|d\psi_c(t)\right\rangle = -\frac{\kappa}{8}\Delta\hat{O}^2(t)\left|\psi_c(t)\right\rangle dt + \frac{\sqrt{\kappa}}{2}\Delta\hat{O}(t)\left|\psi_c(t)\right\rangle dW$$

$$\boxed{\left|d\psi_c(t)\right\rangle = -\frac{\kappa}{8}\left(\hat{O} - \left\langle\hat{O}\right\rangle\right)^2\left|\psi_c(t)\right\rangle dt + \frac{\sqrt{\kappa}}{2}\left(\hat{O} - \left\langle\hat{O}\right\rangle\right)\left|\psi_c(t)\right\rangle dW}$$

"Quantum Trajectory"

"Quantum State Diffusion"

Stochastic Master Equation

$$\left|\psi_c(t+dt)\right\rangle = \hat{K}(t,dt)\left|\psi_c(t)\right\rangle$$

$$= \underbrace{\left(1-\frac{\kappa}{8}\Delta\hat{O}_s^2(t)dt\right)}_{1-\frac{i}{\hbar}\hat{H}_{eff}dt=1-\frac{1}{2}\hat{L}^\dagger\hat{L}dt}\left|\psi_c(t)\right\rangle + \underbrace{\frac{\sqrt{\kappa}}{2}\Delta\hat{O}_s(t)dW}_{\hat{L}dW}\left|\psi_c(t)\right\rangle$$

No "click" "click"

$$d\hat{\rho}_c = d\left(\left|\psi_c(t)\right\rangle\left\langle\psi_c(t)\right|\right) = \left|d\psi_c(t)\right\rangle\left\langle\psi_c(t)\right| + \left|\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right| + \left|d\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right|$$

$$= -\frac{1}{2}\hat{L}^\dagger\hat{L}\,\hat{\rho}_c dt + \frac{1}{2}\hat{\rho}_c\hat{L}^\dagger\hat{L}\,dt + \left(\hat{L}\,\hat{\rho}_c + \hat{\rho}_c\hat{L}\right)dW + \hat{L}\,\hat{\rho}_c\hat{L}^\dagger dW^2$$

$$d\hat{\rho}_c = \underbrace{-\frac{1}{2}\left\{\hat{L}^\dagger\hat{L}\,,\hat{\rho}_c\right\}dt + \hat{L}\,\hat{\rho}_c\hat{L}^\dagger dt}_{Lindbladian} + \underbrace{\left(\hat{L}\,\hat{\rho}_c + \hat{\rho}_c\hat{L}\right)}_{Measurement\ backaction}dW$$

Stochastic Master Equation

$$\left|\psi_c(t+dt)\right\rangle = \hat{K}(t,dt)\left|\psi_c(t)\right\rangle$$

$$= \underbrace{\left(1-\frac{\kappa}{8}\Delta\hat{O}_s^2(t)dt\right)}_{1-\frac{i}{\hbar}\hat{H}_{eff}dt=1-\frac{1}{2}\hat{L}^\dagger\hat{L}dt}\left|\psi_c(t)\right\rangle + \underbrace{\frac{\sqrt{\kappa}}{2}\Delta\hat{O}_s(t)dW}_{\hat{L}dW}\left|\psi_c(t)\right\rangle$$

No "click" "click"

$$d\hat{\rho}_c = d\left(\left|\psi_c(t)\right\rangle\left\langle\psi_c(t)\right|\right) = \left|d\psi_c(t)\right\rangle\left\langle\psi_c(t)\right| + \left|\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right| + \left|d\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right|$$

$$= -\frac{1}{2}\hat{L}^\dagger\hat{L}\,\hat{\rho}_c dt + \frac{1}{2}\hat{\rho}_c\hat{L}^\dagger\hat{L}\,dt + \left(\hat{L}\,\hat{\rho}_c + \hat{\rho}_c\hat{L}\right)dW + \hat{L}\,\hat{\rho}_c\hat{L}^\dagger dW^2$$

$$d\hat{\rho}_c = \underbrace{-\frac{\kappa}{8}\left\{\Delta\hat{O}_s^2,\hat{\rho}_c\right\}dt + \frac{\kappa}{4}\Delta\hat{O}_s\hat{\rho}_c\Delta\hat{O}_s dt}_{Lindbladian} + \underbrace{\frac{\sqrt{\kappa}}{2}\left(\Delta\hat{O}_s\hat{\rho}_c + \hat{\rho}_c\Delta\hat{O}_s\right)dW}_{Measurement\ backaction}$$

Stochastic Master Equation

$$\left|\psi_c(t+dt)\right\rangle = \hat{K}(t,dt)\left|\psi_c(t)\right\rangle$$

$$= \underbrace{\left(1-\frac{\kappa}{8}\Delta\hat{O}_s^2(t)dt\right)}_{1-\frac{i}{\hbar}\hat{H}_{eff}dt=1-\frac{1}{2}\hat{L}^\dagger\hat{L}dt}\left|\psi_c(t)\right\rangle + \underbrace{\frac{\sqrt{\kappa}}{2}\Delta\hat{O}_s(t)dW}_{\hat{L}dW}\left|\psi_c(t)\right\rangle$$

$$d\hat{\rho}_c = d\left(\left|\psi_c(t)\right\rangle\left\langle\psi_c(t)\right|\right) = \left|d\psi_c(t)\right\rangle\left\langle\psi_c(t)\right| + \left|\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right| + \left|d\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right|$$

$$= -\frac{1}{2}\hat{L}^\dagger\hat{L}\,\hat{\rho}_c dt + \frac{1}{2}\hat{\rho}_c\hat{L}^\dagger\hat{L}\,dt + \left(\hat{L}\,\hat{\rho}_c + \hat{\rho}_c\hat{L}\right)dW + \hat{L}\,\hat{\rho}_c\hat{L}^\dagger dW^2$$

$$d\hat{\rho}_c = \underbrace{-\frac{\kappa}{8}\left[\hat{O}_s,\left[\hat{O}_s,\hat{\rho}_c\right]\right]dt}_{Lindbladian} + \underbrace{\frac{\sqrt{\kappa}}{2}\left(\Delta\hat{O}_s\hat{\rho}_c + \hat{\rho}_c\Delta\hat{O}_s\right)dW}_{Measurement\ backaction}$$

Stochastic Master Equation

$$\left|\psi_c(t+dt)\right\rangle = \hat{K}(t,dt)\left|\psi_c(t)\right\rangle$$

$$= \underbrace{\left(1-\frac{\kappa}{8}\Delta\hat{O}_s^2(t)dt\right)}_{1-\frac{i}{\hbar}\hat{H}_{eff}dt = 1-\frac{1}{2}\hat{L}^\dagger\hat{L}dt}\left|\psi_c(t)\right\rangle + \underbrace{\frac{\sqrt{\kappa}}{2}\Delta\hat{O}_s(t)dW}_{\hat{L}dW}\left|\psi_c(t)\right\rangle$$

$$d\hat{\rho}_c = d\left(\left|\psi_c(t)\right\rangle\left\langle\psi_c(t)\right|\right) = \left|d\psi_c(t)\right\rangle\left\langle\psi_c(t)\right| + \left|\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right| + \left|d\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right|$$

$$= -\frac{1}{2}\hat{L}^\dagger\hat{L}\,\hat{\rho}_c dt + \frac{1}{2}\hat{\rho}_c\hat{L}^\dagger\hat{L}\,dt + \left(\hat{L}\,\hat{\rho}_c + \hat{\rho}_c\hat{L}\right)dW + \hat{L}\,\hat{\rho}_c\hat{L}^\dagger dW^2$$

$$d\hat{\rho}_c = -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}_c\right] + \mathcal{L}_{Decoherence}[\hat{\rho}_c]$$

General SME

$$-\frac{\kappa}{8}\left[\hat{O}_s,\left[\hat{O}_s,\hat{\rho}_c\right]\right]dt + \frac{\sqrt{\kappa}}{2}\left(\Delta\hat{O}_s\hat{\rho}_c + \hat{\rho}_c\Delta\hat{O}_s\right)dW$$

Stochastic Master Equation

$$\left|\psi_c(t+dt)\right\rangle = \hat{K}(t,dt)\left|\psi_c(t)\right\rangle$$

$$= \underbrace{\left(1-\frac{\kappa}{8}\Delta\hat{O}_s^2(t)dt\right)}_{1-\frac{i}{\hbar}\hat{H}_{eff}dt = 1-\frac{1}{2}\hat{L}^\dagger\hat{L}dt}\left|\psi_c(t)\right\rangle + \underbrace{\frac{\sqrt{\kappa}}{2}\Delta\hat{O}_s(t)dW}_{\hat{L}dW}\left|\psi_c(t)\right\rangle$$

$$d\hat{\rho}_c = d\left(\left|\psi_c(t)\right\rangle\left\langle\psi_c(t)\right|\right) = \left|d\psi_c(t)\right\rangle\left\langle\psi_c(t)\right| + \left|\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right| + \left|d\psi_c(t)\right\rangle\left\langle d\psi_c(t)\right|$$

$$= -\frac{1}{2}\hat{L}^\dagger\hat{L}\,\hat{\rho}_c dt + \frac{1}{2}\hat{\rho}_c\hat{L}^\dagger\hat{L}\,dt + \left(\hat{L}\,\hat{\rho}_c + \hat{\rho}_c\hat{L}\right)dW + \hat{L}\,\hat{\rho}_c\hat{L}^\dagger dW^2$$

$$\boxed{\begin{aligned} d\hat{\rho}_c &= -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}_c\right] + \mathcal{L}_{Decoherence}[\hat{\rho}_c] \\ &\quad -\frac{\kappa}{8}\left[\hat{O}_s,\left[\hat{O}_s,\hat{\rho}_c\right]\right]dt + \frac{\sqrt{\kappa}}{2}\left(\Delta\hat{O}_s\hat{\rho}_c + \hat{\rho}_c\Delta\hat{O}_s\right)dW \end{aligned}}$$

General SME

QND Measurement via continuous measurement

Quantum Nondemolition (QND) Measurement

- Standard paradigm of quantum measurement.
- If a system is in an eigenstate, measurement doesn't disturb it; backaction into conjugate variables.
- QND measurement can reduce uncertainty in observable (squeezing)

- Mathematical definition:

QND measurement of an observable ⇔ Kraus operator commutes observable

$$\left[\hat{K}_{\mathcal{M}},\hat{O}\right]=0$$

QND Measurement via continuous measurement

Evolution of moments of observables under continuous measurement:

$$d\left\langle \hat{O}^n \right\rangle = Tr\left(\hat{O}^n \, d\hat{\rho}_c\right) = \sqrt{\kappa}\left\langle \hat{O}^n \Delta\hat{O} \right\rangle dW$$

$$\Rightarrow d\left\langle \hat{O} \right\rangle = \sqrt{\kappa}\left\langle \hat{O}\Delta\hat{O} \right\rangle dW = \sqrt{\kappa}\left\langle \Delta\hat{O}^2 \right\rangle dW$$

$$\Rightarrow d\left\langle \hat{O}^2 \right\rangle = \sqrt{\kappa}\left\langle \hat{O}^2\Delta\hat{O} \right\rangle dW$$

$$\Rightarrow d\left\langle \Delta\hat{O}^2 \right\rangle = d\left(\left\langle \hat{O}^2 \right\rangle - \left\langle \hat{O} \right\rangle^2\right) = d\left\langle \hat{O}^2 \right\rangle - 2\left\langle \hat{O} \right\rangle d\left\langle \hat{O} \right\rangle - \left(d\left\langle \hat{O} \right\rangle\right)^2$$

$$= \sqrt{\kappa}\left(\left\langle \hat{O}^2\Delta\hat{O} \right\rangle - 2\left\langle \hat{O} \right\rangle\left\langle \Delta\hat{O}^2 \right\rangle\right) dW - \kappa\left\langle \Delta\hat{O}^2 \right\rangle^2 dt$$

$$= \sqrt{\kappa}\left\langle \Delta\hat{O}^3 \right\rangle dW - \kappa\left\langle \Delta\hat{O}^2 \right\rangle^2 dt$$

0 Gaussian statistics

QND Measurement via continuous measurement

Evolution of moments of observables under continuous measurement:

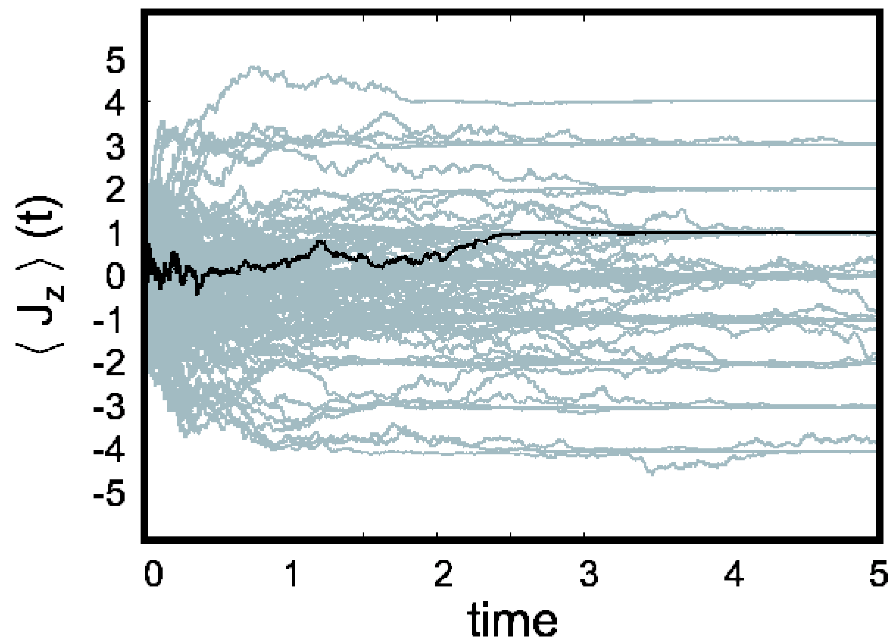
$$d\left\langle \hat{O} \right\rangle = \sqrt{\kappa}\left\langle \Delta\hat{O}^2 \right\rangle dW$$

Mean position randomly kicked (projection noise)

$$d\left\langle \Delta\hat{O}^2 \right\rangle = -\kappa\left\langle \Delta\hat{O}^2 \right\rangle^2 dt \Rightarrow \left\langle \Delta\hat{O}^2(T) \right\rangle = \frac{\left\langle \Delta\hat{O}^2(0) \right\rangle}{1+\kappa T} \Rightarrow \frac{\left\langle \Delta\hat{O}^2(0) \right\rangle}{\kappa T} \quad \kappa T >> 1$$

Projection noise squeezed to the shot noise resolution of probe

Example:
QND measurement of spin projection J_z for spin J=4



Stockton *et al.*, Phys. Rev. A **70**, 002206 (2004)

Take-Away Lessons

• Von Neumann paradigm of quantum measurement describes how a quantum meter gains information about a quantum system.

• Most general measurement in quantum mechanics: POVM
- Arbitrary number of possible measurement outcomes.
- Born rule: probability of measurement outcome.
- Measurement back action: Kraus operators update state conditioned on measurement outcome.

• "Weak" nonprojective measurements
- Different projective outcomes not fully resolvable by the meter.
- Measurement backaction weakly disturbs the state.
- Information gain / disturbance tradeoff.

• Continuous measurement
- Continuous probe "slice of time" differentially measures system
- Stochastic Schrödinger equation: continuous-time evolution of the state conditioned the continuous measurement record